Computing & Microfluidics Lab
國立成功大學工程科學系
計算與微流體晶片 實驗室

Adaptive CESE Method for Solving Unsteady Euler Equations

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To Tony Sheu

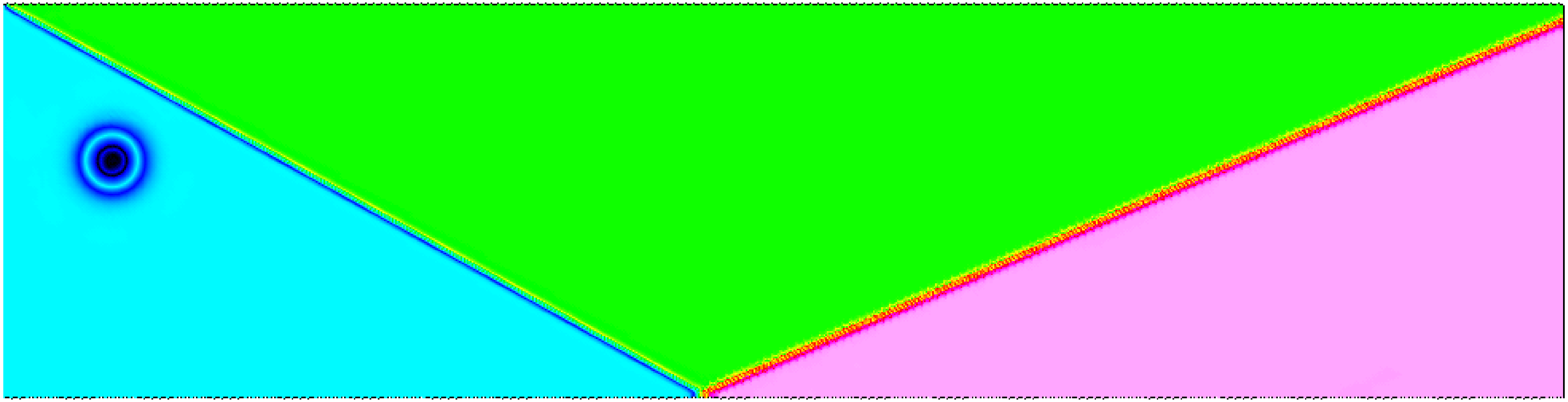
Wishing you

Good spirit, Good health,

Keep random walk with deterministic mind

Enjoy life!

Introduction



Shock Vortex Interaction

Flow discontinuities, Unsteady waves,

Accuracy, Easy treatment of BCs, Length Scales

NASA Glenn Research Center website:
http://www.grc.nasa.gov/WWW/microbus/

The Inventor of the CESE Method

Dr. Sin-Chung Chang
NASA Glenn Research Center

Chang SC, (1995), The method of space-time conservation element and solution element– a new approach for solving the Navier-Stokes and Euler equations, *Journal of Computational Physics*, 119, 295-324.

CESE Method

(1) The method is a complete explicit scheme.

(2) Non-dissipative scheme.

(3) Add numerical dissipation as desired.

(4) Enforced space-time flux conservation, local/global flux conservation.

(5) The spatial derivative $\frac{\partial u}{\partial x}$ is treated as unknown variable w/o discretization.

(6) No flux reconstruction at mesh interface (Riemann problem).

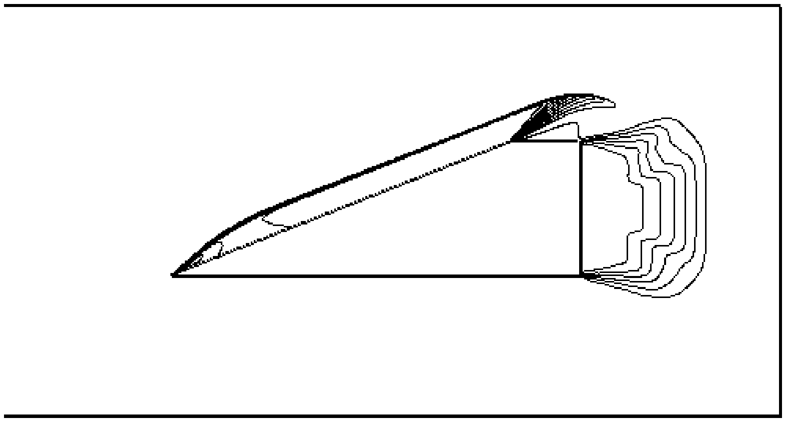
Our Former CESE Applications
Shock Diffraction

Shock Diffraction
over a 75 degrees corner
(M_0=2.5)
Density Contours

CE/SE Group
NCKU ES Computing & Microfluidics Lab.

Tseng T and Yang RJ 2005, *Shock Waves,* Vol. 14, 307-311

Our Former CESE Applications Supersonic Flow over a Wedge

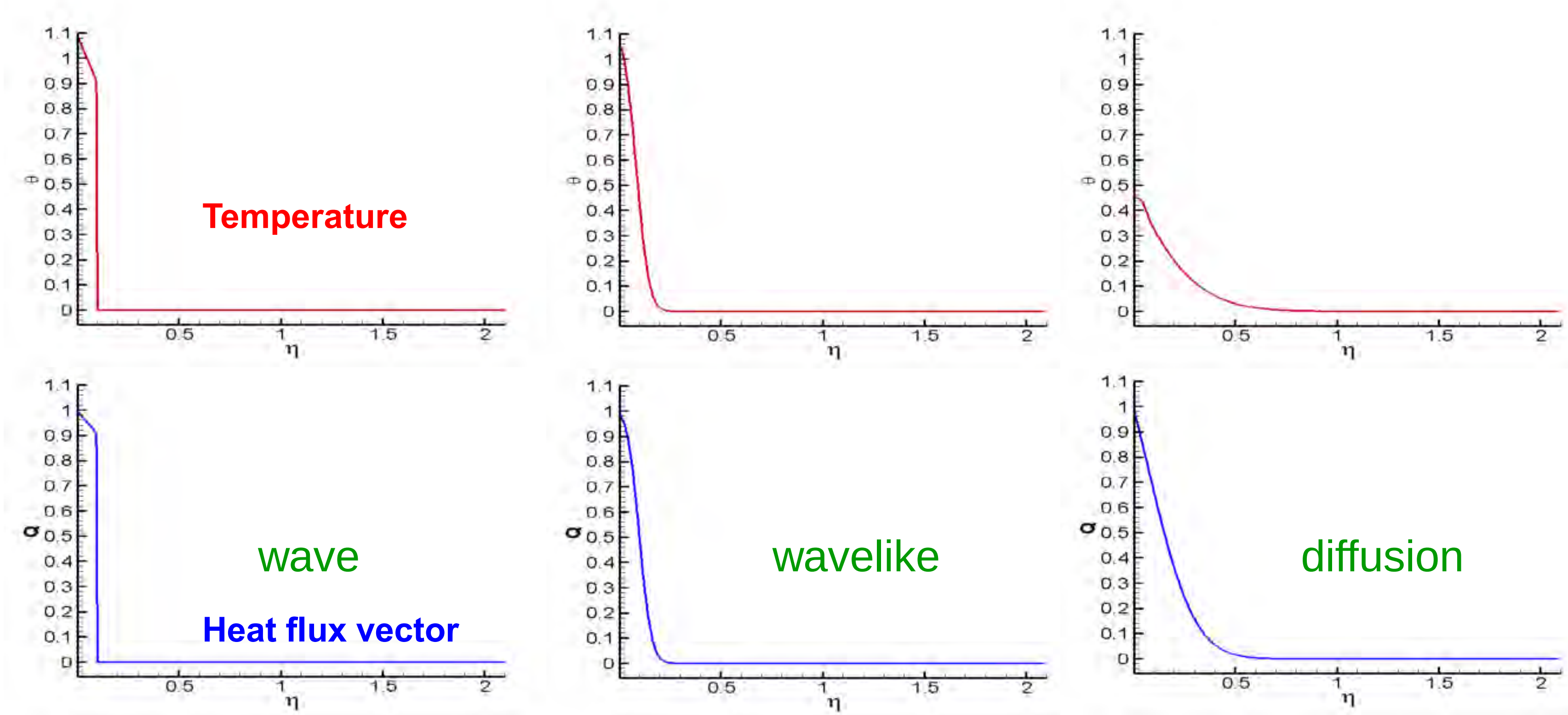


Tseng T and Yang RJ 2006, *AIAA Journal*, Vol.44, 1040-1047

1-D Thermal Waves

Presentation of wave, wavelike and diffusion behaviors

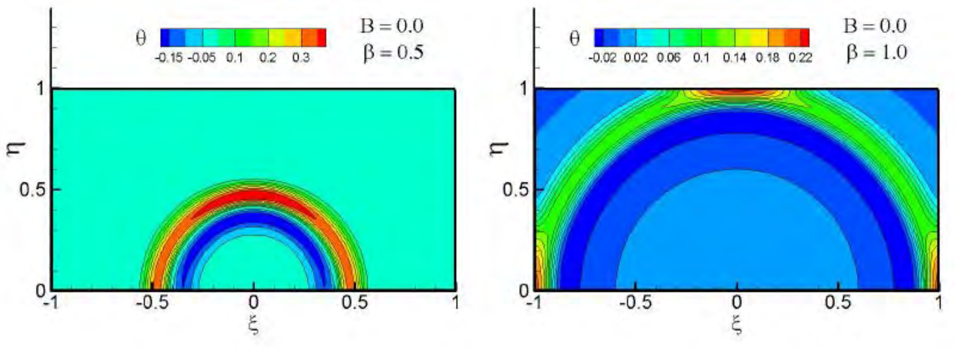
CV model **DPL model** **Diffusion**



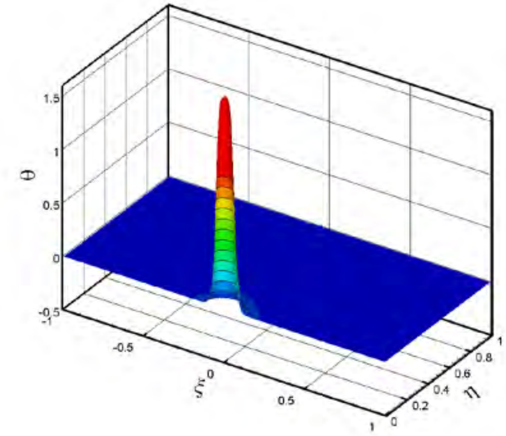
Chou Y and Yang RJ, 2008, International Journal of Heat and Mass Transfer, Vol. 51, 3525-3534

2D Thermal Wave

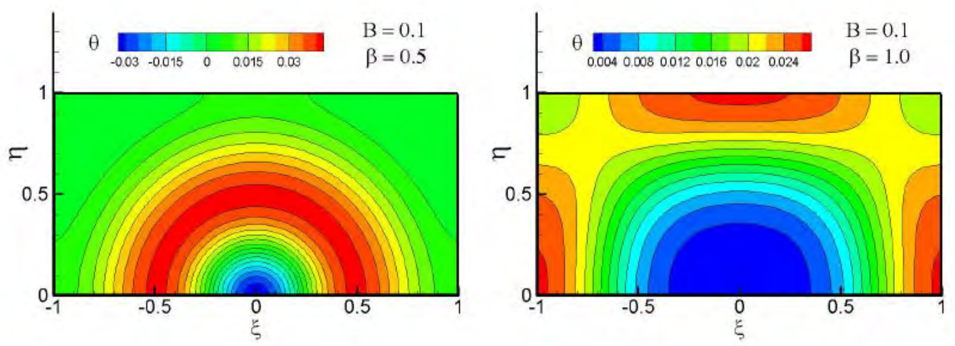
Wave behavior in the condition of B=0.0



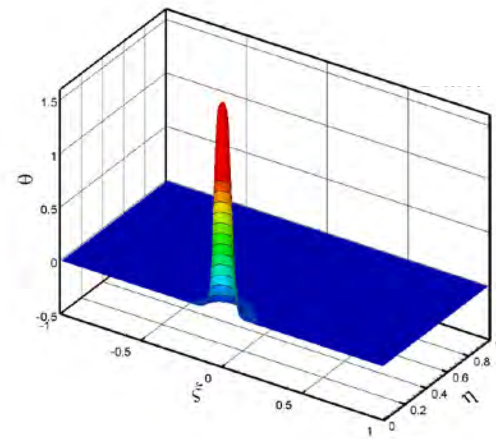
(a) β = 0.5 (b) β = 1.0



Wavelike behavior in the condition of B=0.1



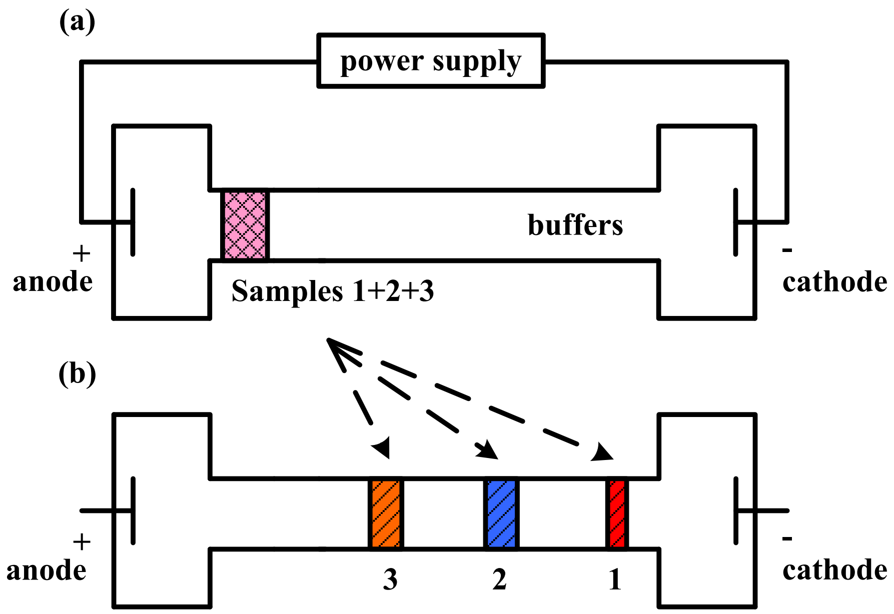
(a) β = 0.5 (b) β = 1.0



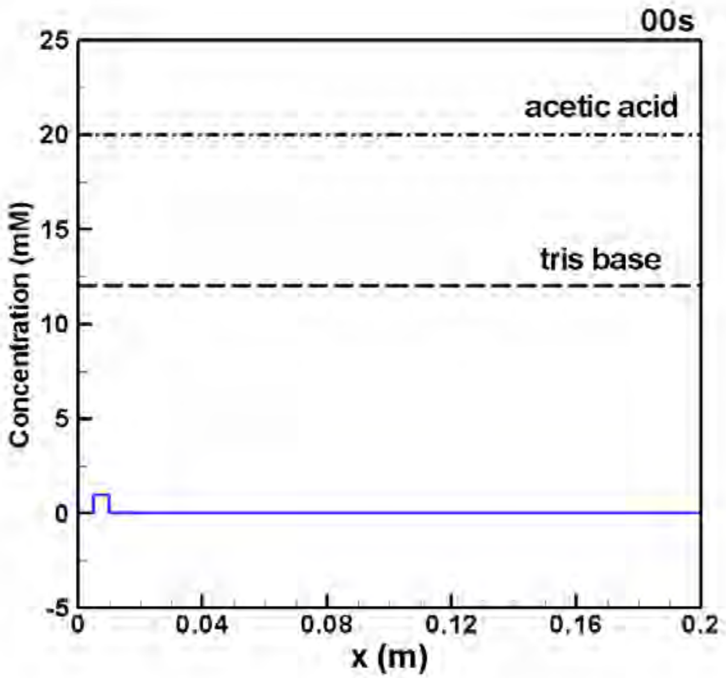
Chou Y and Yang RJ, 2009, International Journal of Heat and Mass Transfer, Vol.52, 239-249

Zone Electrophoresis

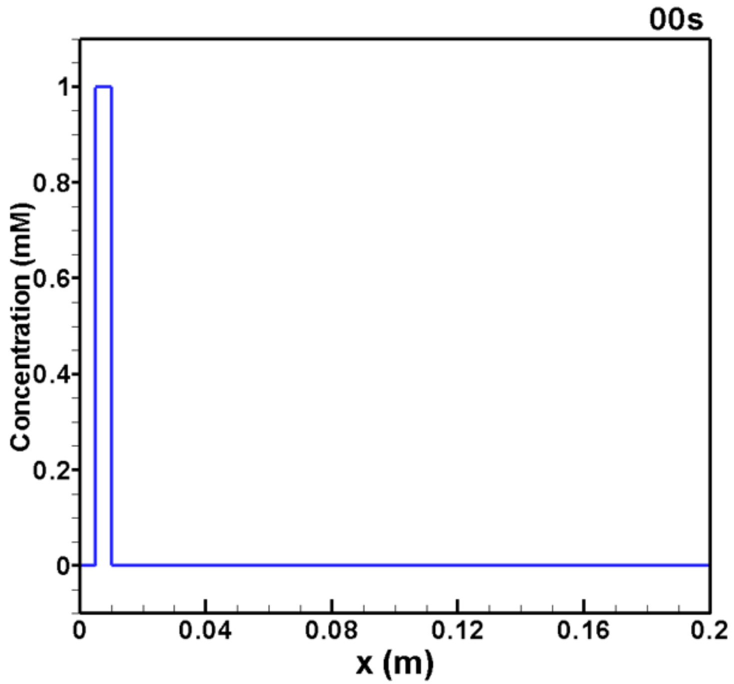
Introduced in the 1960s, the technique of capillary zone electrophoresis (CZE) was designed to separate species based on their size to charge ratio in the interior of a small capillary filled with an electrolyte.



Validation of CESE scheme in ZE



2 μA, Δx=25μm, 8001 grids

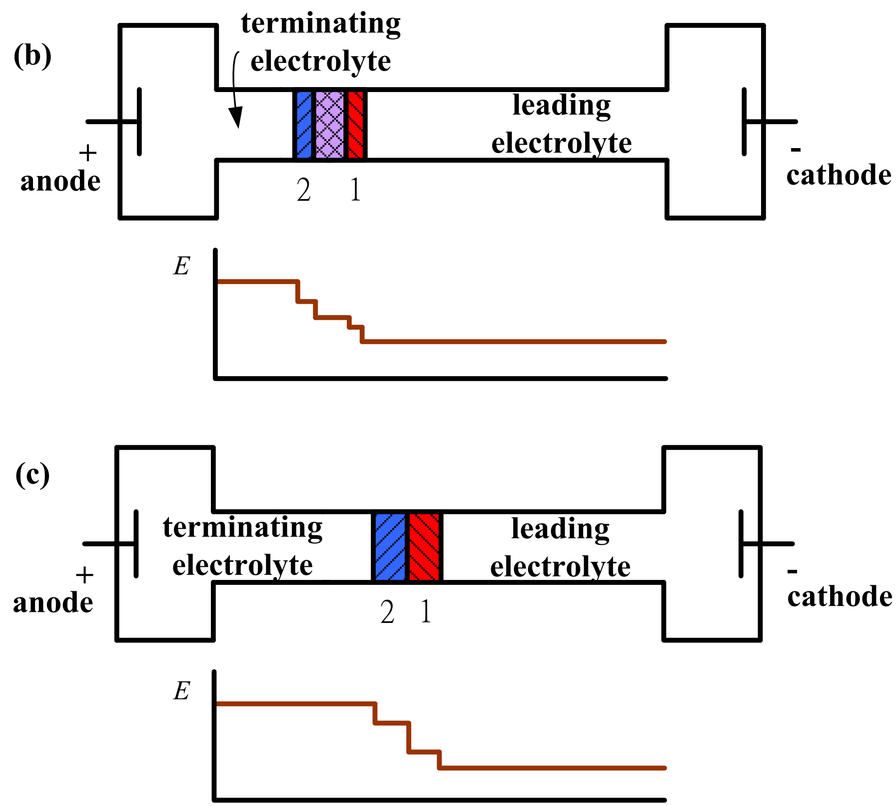


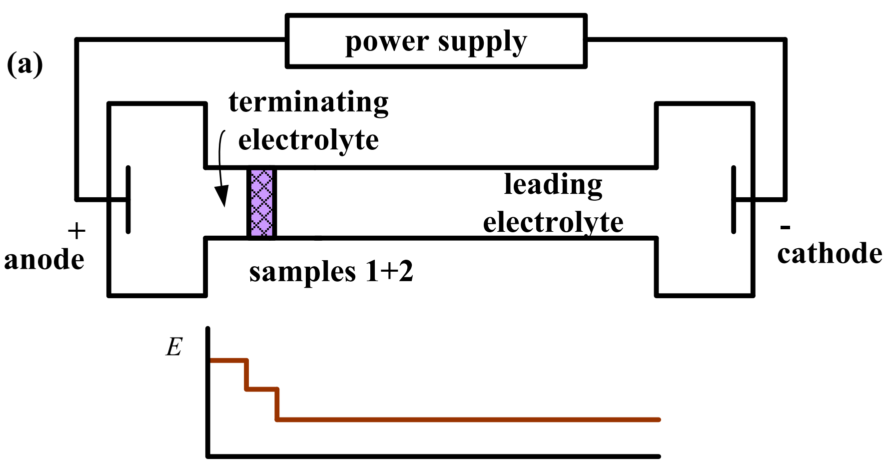
Yu JW, Chou Y and Yang RJ, 2008, *Electrophoresis*, Vol.29, 1048-1057

IsoTachoPhoresis (ITP)

- **Isotachophoresis (ITP)** (Greek: *iso* = equal, *tachos* = speed, *phoresis* = migration) is a technique in analytical chemistry used to separate charged particles.

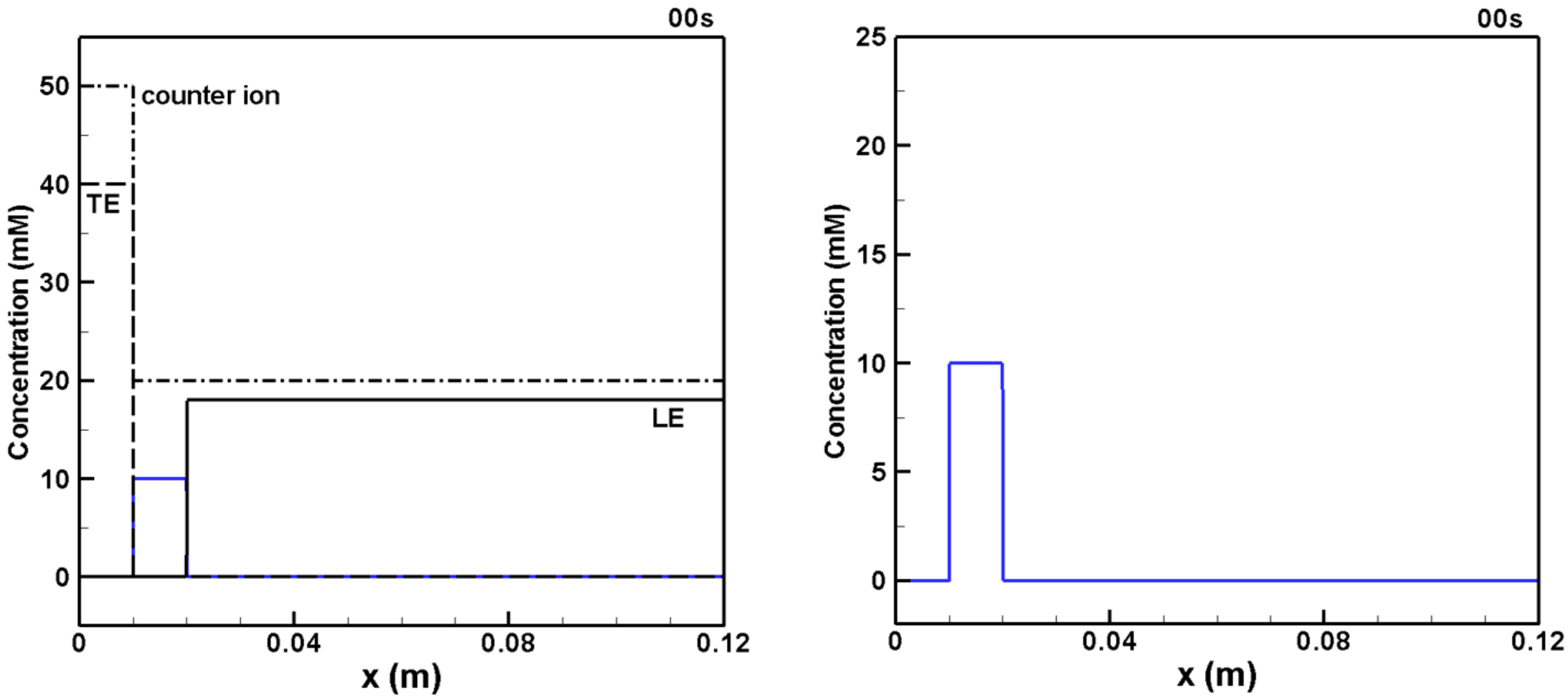
- In isotachophoresis the sample is introduced between a fast leading electrolyte and a slow terminating electrolyte.





Validation of CESE scheme in ITP

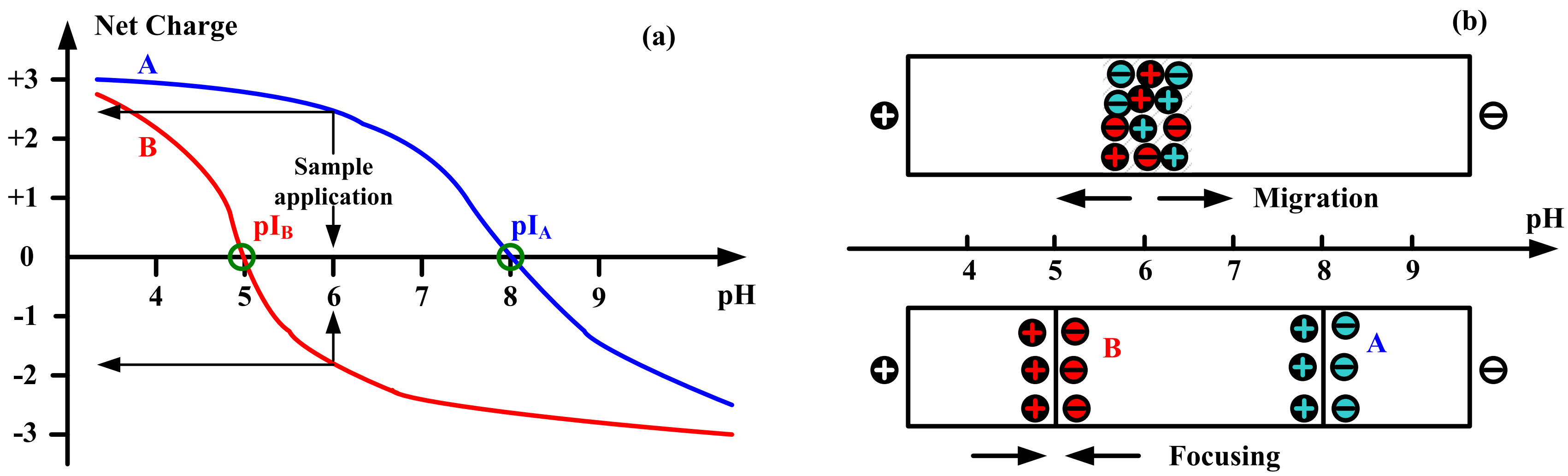
Δx=100μm, 1201 grids



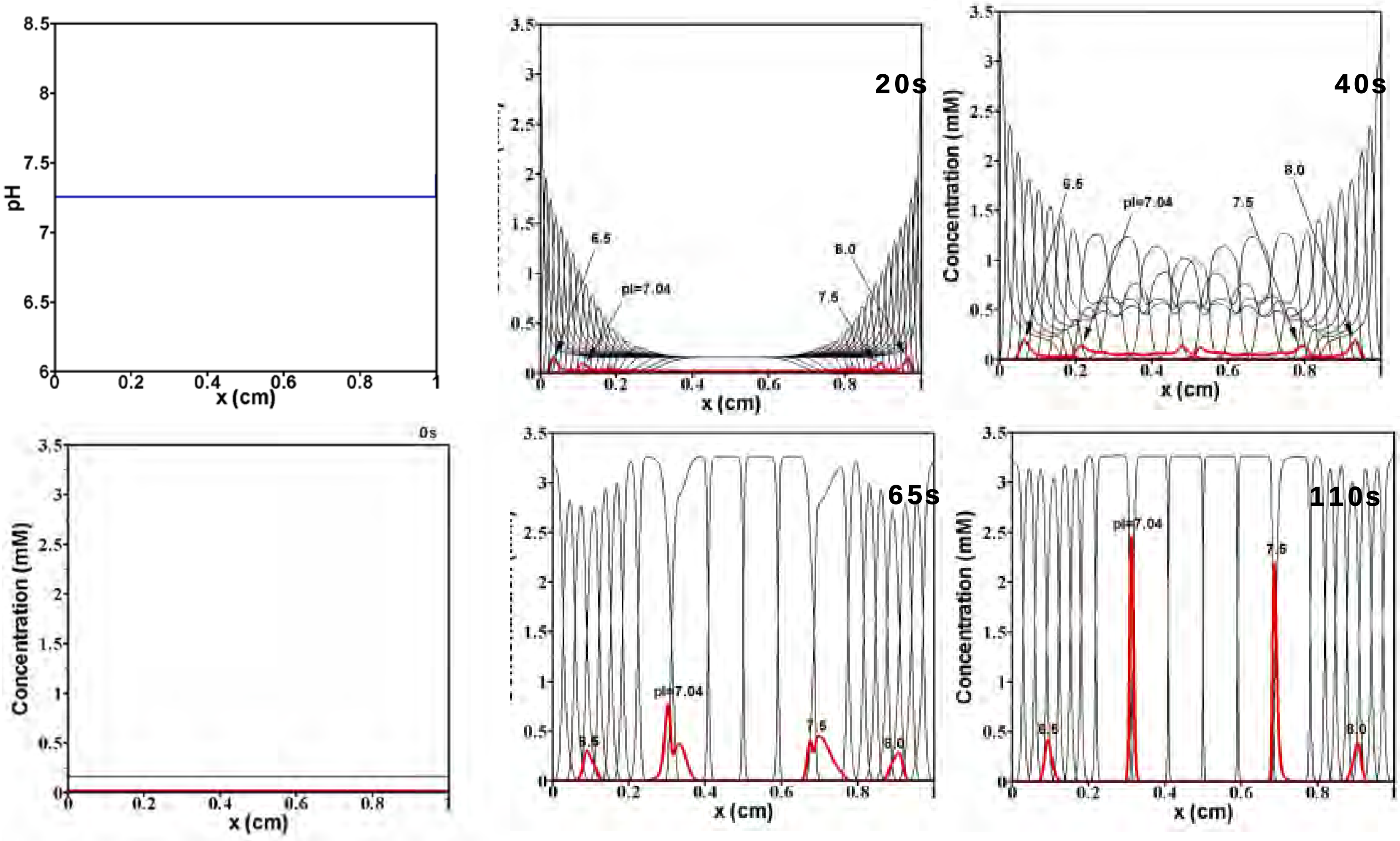
Chou Y and Yang RJ, 2009, *Electrophoresis*, Vol.30, 819-830

Isoelectric Focusing (IEF)

- The pH at which charge reversal occurs, i.e. where the net charge is zero, is called the isoelectric point, or *pI*.



IEF results- transition solutions



Chou Y and Yang RJ, 2010 *Journal of Chromatography A,* Vol. 1217, 394-404

Motivation

(1)Applying a fine mesh to solve complex fluid flow problems is computationally expensive.

(2) Adaptive mesh provides suitable mesh as necessary to save computational cost.

(3) The solutions in the current time-step are computed directly from the solutions obtained in the previous time-step without the need for extrapolation or interpolation.

Traditional CESE Method (Stationary Mesh)

Example: $\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0$

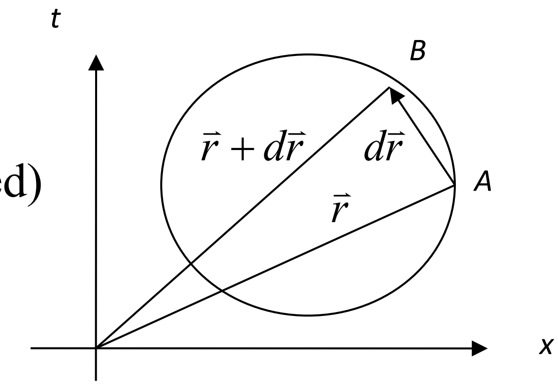
(a is constant speed)

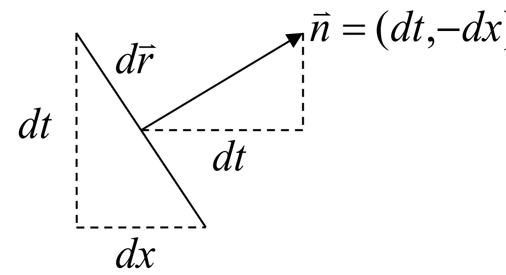
$\vec{h} = (au, u)$

$$\nabla \cdot \vec{h} = \frac{\partial}{\partial x}(au) + \frac{\partial}{\partial t}(u) = 0$$

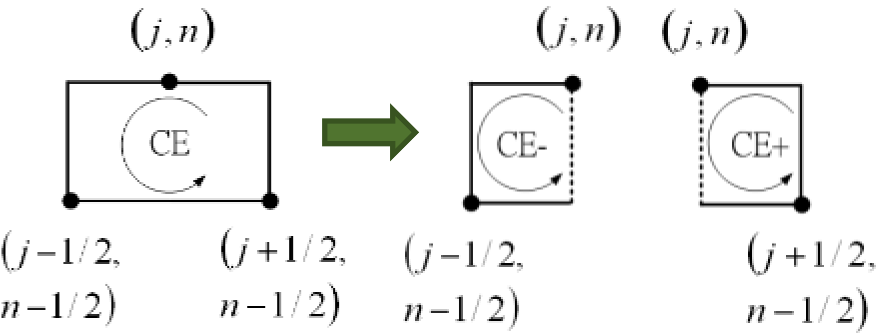
$$\int_v \nabla \cdot \vec{h} \cdot dv = \int_S \vec{h} \cdot \vec{n} \cdot d\sigma =$$

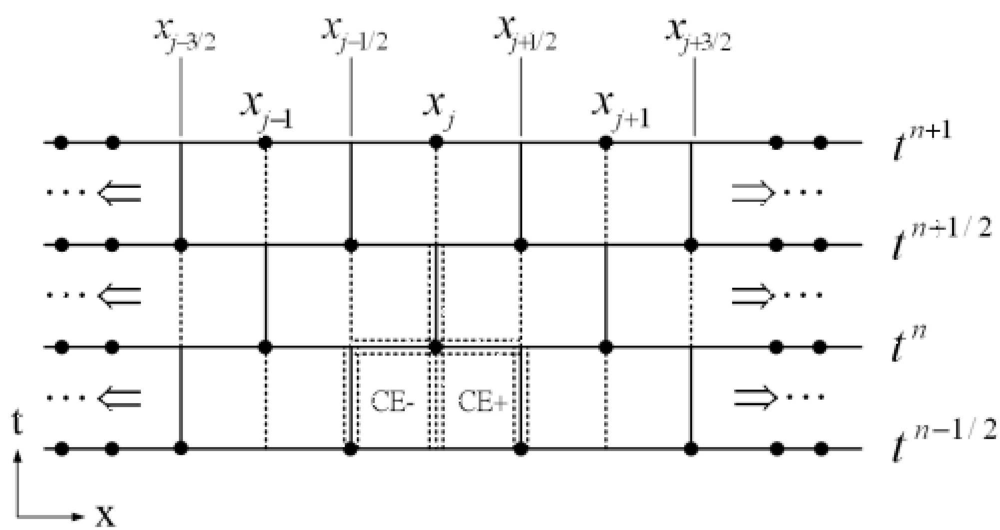
$$\int_s (au, u) \cdot (dt, -dx) = \int_s au \cdot dt - \int_s u \cdot dx = 0$$





A surface element $d\mathbf{s}$ and a line segment $d\mathbf{r}$ on the boundary $S(V)$.





Two Eqs. → Solve u and u_x.

Introduction of one-dimensional adaptive CESE method

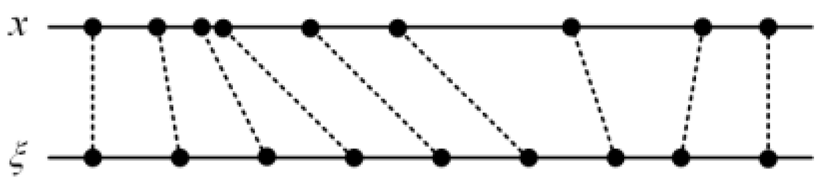
Mesh redistribution

Physical Coordinate: $\mathbf{x} = (x_1, x_2, \ldots\ldots, x_d)$

Computational Coordinate: $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots\ldots, \xi_d)$

Quasi-Static Equidistribution Principle

$$\left(\omega x_{\xi}\right)_{\xi} = 0$$



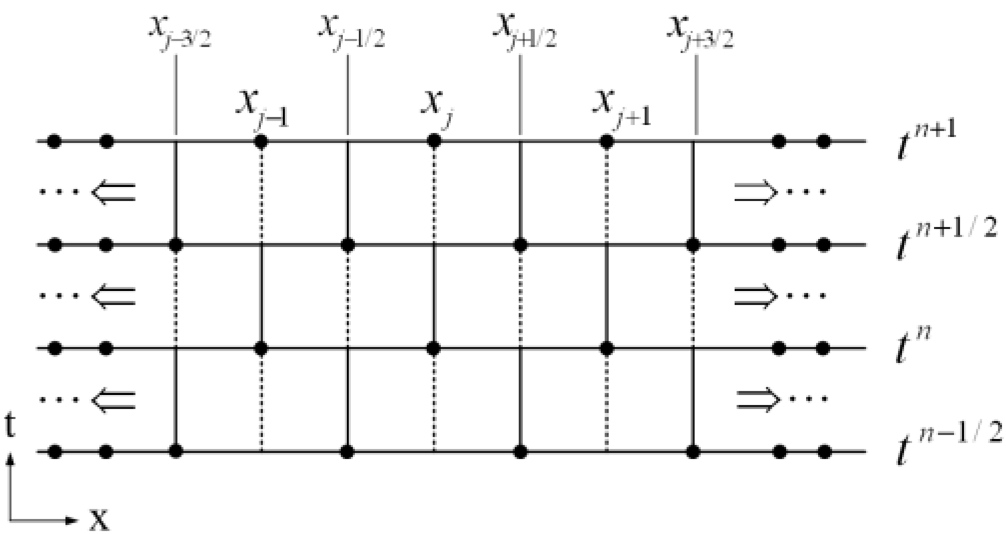
Monitor Function: $\omega = \sqrt{1 + \beta\left(\nabla\Phi\right)^2}$

Φ: Flow variables

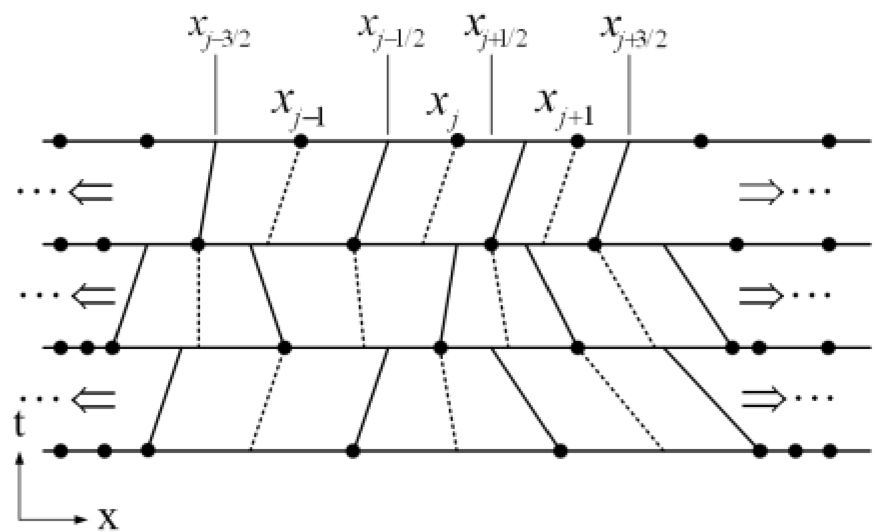
β : scaling parameter

Space-time domain

Stationary



Moving



The space-time is divided into non-overlapped rectangular regions as the conservation elements and each mesh nodes (marked by filled circle) are setting as the solution elements.

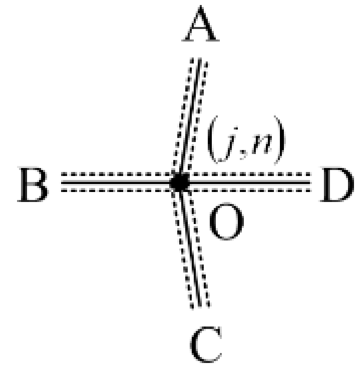
The solution element (SE) and conservation element (CE)

Flow properties are assumed continuous within SE. For any (x, y) $\in$ SE(j, n), $u_m(x, t)$ and $f_m(x, t)$ are approximated by $u^*_m(x, t)$ and $f^*_m(x, t)$.

They are defined as :

$$u_m^*(x,t;j,n) = (u_m)_j^n + (u_{mx})_j^n (x - x_j) + (u_{mt})_j^n (t - t^n)$$

$$f_m^*(x,t;j,n) = (f_m)_j^n + (f_{mx})_j^n (x - x_j) + (f_{mt})_j^n (t - t^n)$$

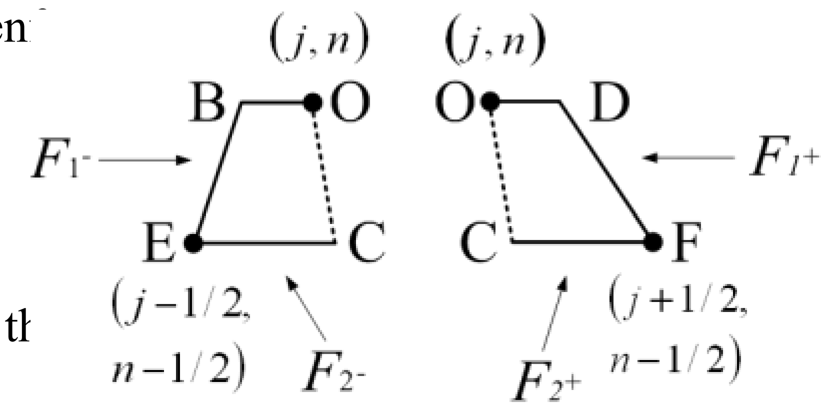


Solution element in point (j,n)

The CE combines two basic CEs, CE+ and CE-, and to en
flux conservation across CE surfaces we have

$$(u_m)_j^n = -(F_{1^-} + F_{2^-} + F_{1^+} + F_{2^+})/(2\Delta x_3),\ m = 1,2,3$$

where the F_{1^-} , F_{2^-} , F_{2^+} and F_{1^+} denote the fluxes of cross th
line segment BE ,EC ,CF and FD, respectively.



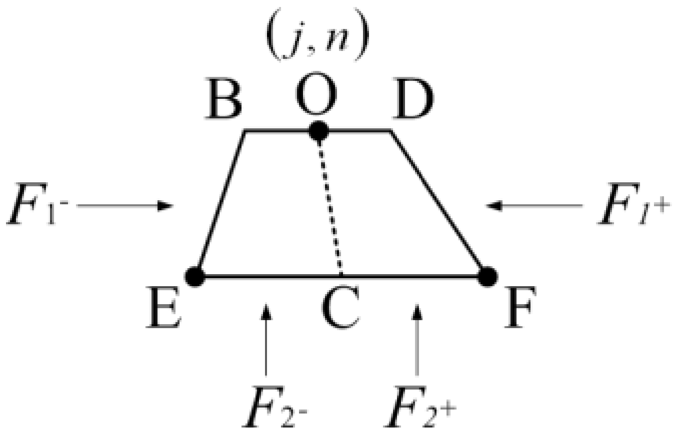
Basic conservation element CE+ and CE-

(1) $F_{1^-} = \oint_{\Gamma(B\to E)} \mathbf{h}^* \cdot d\mathbf{s}$

$= \varphi_m(E) - \varphi_m(B)$

$= -\Big[(f_m)_{j-1/2}^{n-1/2}(\Delta t/2) - (u_m)_{j-1/2}^{n-1/2}(\Delta x_1^-) + (f_{mt})_{j-1/2}^{n-1/2}(\Delta t/2)^2/2$

$-(u_{mx})_{j-1/2}^{n-1/2}(\Delta x_1^-)^2/2 + (f_{mx})_{j-1/2}^{n-1/2}(\Delta x_1^-)(\Delta t/2)\Big]$



The compounded conservation elements

(2) $F_{2^-} = \oint_{\Gamma(E\to C)} \mathbf{h}^* \cdot d\mathbf{s}$

$= \varphi_m(C) - \varphi_m(E)$

$= -(u_m)_{j-1/2}^{n-1/2}(\Delta x_2^-) - (u_{mx})_{j-1/2}^{n-1/2}(\Delta x_2^-)^2/2$

(3) $F_{2^+} = \oint_{\Gamma(C\to F)} \mathbf{h}^* \cdot d\mathbf{s}$

$= \varphi_m(F) - \varphi_m(C)$

$= -\Big[-(u_m)_{j+1/2}^{n-1/2}(\Delta x_2^+) - (u_{mx})_{j+1/2}^{n-1/2}(\Delta x_2^+)^2/2\Big]$

(4) $F_{1^+} = \oint_{\Gamma(F\to D)} \mathbf{h}^* \cdot d\mathbf{s}$

$= \varphi_m(D) - \varphi_m(F)$

$= (f_m)_{j+1/2}^{n-1/2}(\Delta t/2) - (u_m)_{j+1/2}^{n-1/2}(\Delta x_1^+) + (f_{mt})_{j+1/2}^{n-1/2}(\Delta t/2)^2/2$

$-(u_{mx})_{j+1/2}^{n-1/2}(\Delta x_1^+)^2/2 + (f_{mx})_{j+1/2}^{n-1/2}(\Delta x_1^+)(\Delta t/2)$

The $\left(u'_m\right)^n_{j\pm1/2}$ is the first-order Taylor's series approximation of u_m at $(j\pm1/2,n)$.

$$\left(u'_m\right)^n_{j\pm1/2} = \left(u_m\right)^{n-1/2}_{j\pm1/2} + \left(u_{mx}\right)^{n-1/2}_{j\pm1/2}\left(\Delta x_1^{\pm}\right) + \left(u_{mt}\right)^{n-1/2}_{j\pm1/2}\left(\Delta t/2\right),\ m=1,2,3,$$

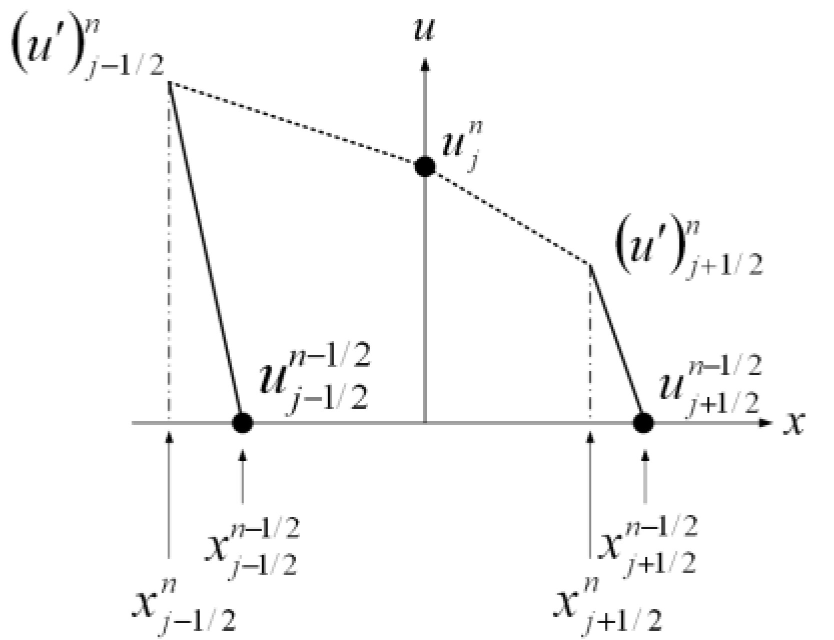
The $\left(\hat{u}_{mx^{\pm}}\right)^n_j$ denote first-order spatial derivatives of the backward and forward differential schemes at point (j,n).

$$\left(\hat{u}_{mx^{\pm}}\right)^n_j = \frac{\left(u'_m\right)^n_{j\pm1/2} - \left(u_m\right)^n_j}{x^n_{j\pm1/2} - x^n_j},\ m=1,2,3$$

The solution gradient $\left(\hat{u}_{mx}\right)^n_j$ are evaluated by

$$\left(\hat{u}_{mx}\right)^n_j = \frac{\left|\left(\hat{u}_{mx^+}\right)^n_j\right|^{\alpha}\left(\hat{u}_{mx^-}\right)^n_j + \left|\left(\hat{u}_{mx^-}\right)^n_j\right|^{\alpha}\left(\hat{u}_{mx^+}\right)^n_j}{\left|\left(\hat{u}_{mx^+}\right)^n_j\right|^{\alpha} + \left|\left(\hat{u}_{mx^-}\right)^n_j\right|^{\alpha}}$$

$$\left(\left|\left(\hat{u}_{mx^+}\right)^n_j\right| + \left|\left(\hat{u}_{mx^-}\right)^n_j\right| > 0\right)$$



Schematic of weighting function

Adaptive-CESE Algorithm

(1) Give the initial solutions, $(u_m)_j^0$ and $(u_{mx})_j^0$ with a uniform mesh to the physical domain at time $t = 0$.

(2) Compute and smooth the monitor function values $\omega_{j+1/2}$.

(3) Move each grid point from its current location x_j^{μ} to a new location $x_j^{\mu+1}$ using the Gauss-Seidel iteration method.

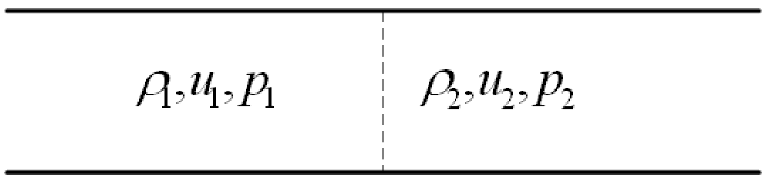
(4) Compute the flow variables, $(u_m)_j^n$ and $(u_{mx})_j^n$ using the adaptive CESE scheme.

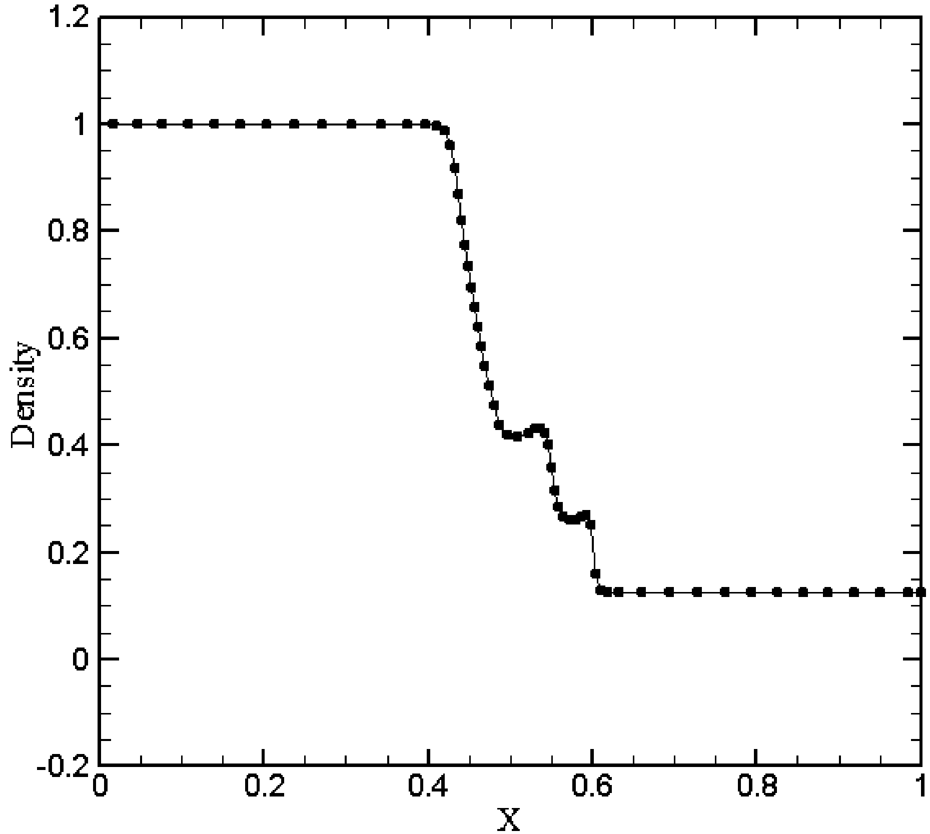
(5) If $t^{n+1/2} < T$ output time, set $(u_m)_j^0 = (u_m)_j^{n+1/2}$ and $x_j^0 = \left(x_{j+1/2}^{n+1/2} + x_{j-1/2}^{n+1/2}\right)/2$ and then return to step (2) for the next time step.

Shock tube problems (1)

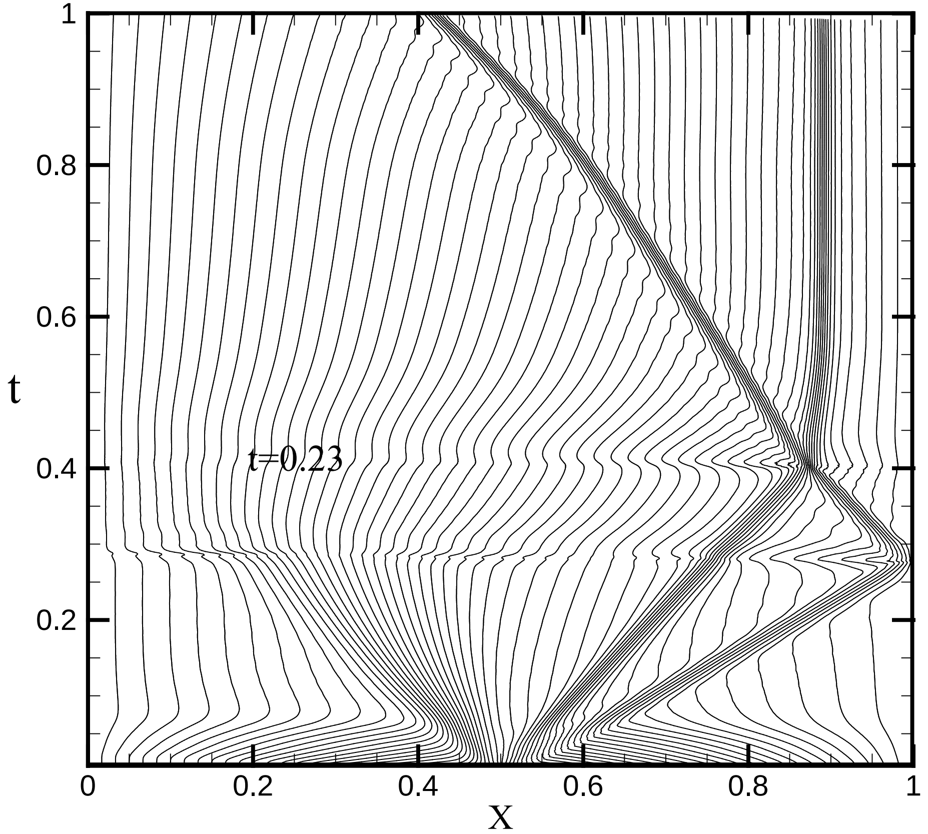
(1)The Sod’s shock tube proble

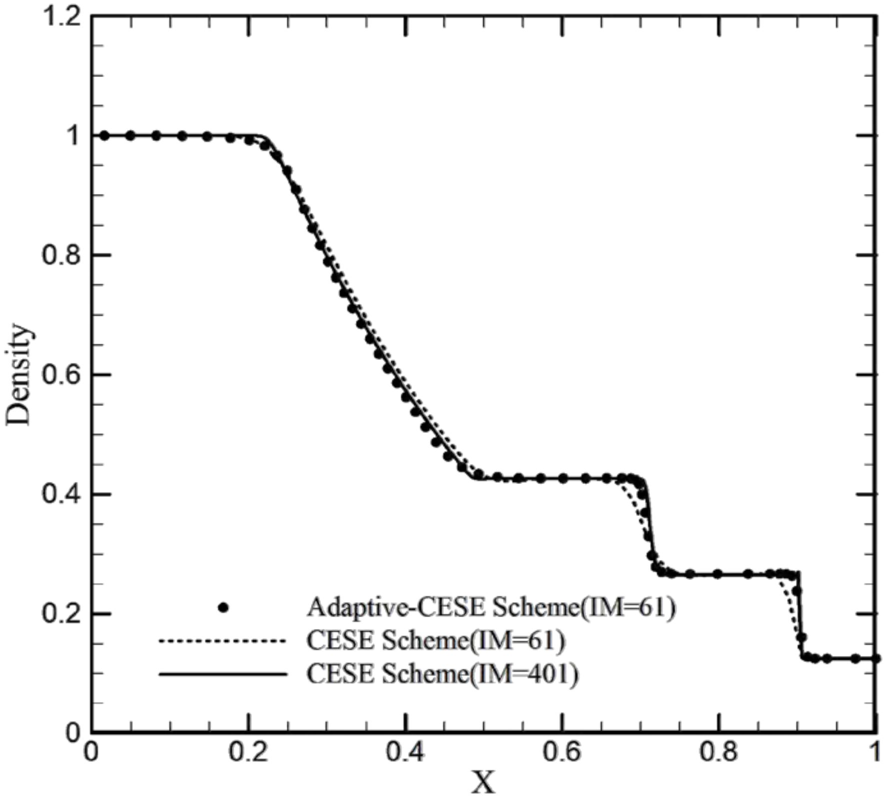
$$(\rho, u, p) = \begin{cases} (1.0, 0, 1.0) & \text{if } x \le 0.5 \\ (0.125, 0, 0.1) & \text{if } x \ge 0.5 \end{cases}$$





Shock tube problems (1)

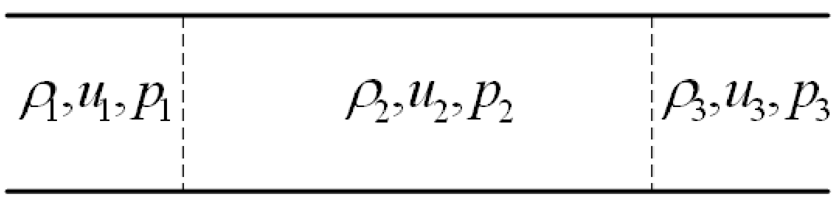


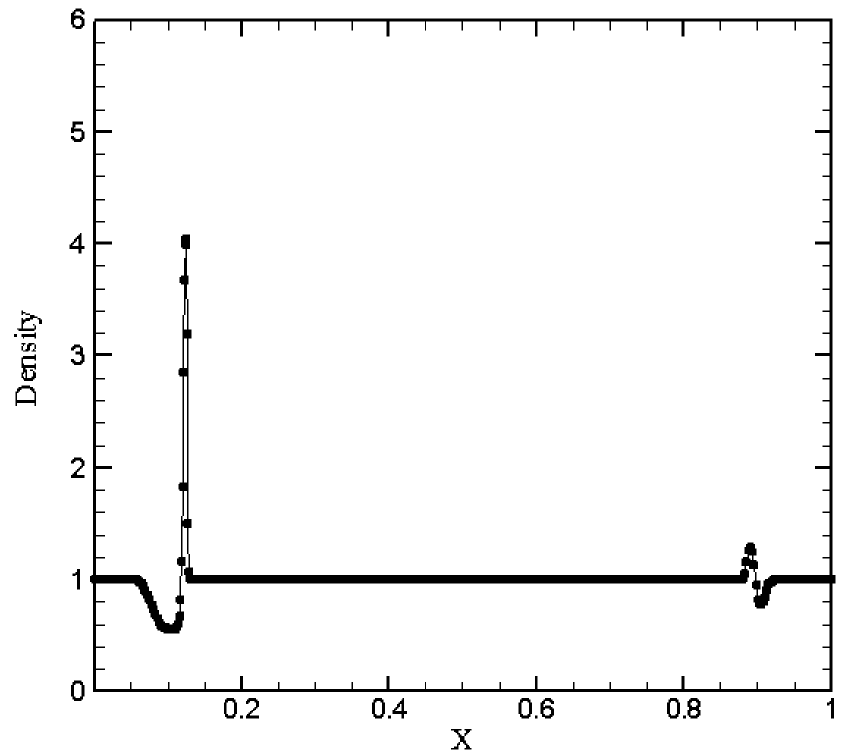


Shock tube problems (2)

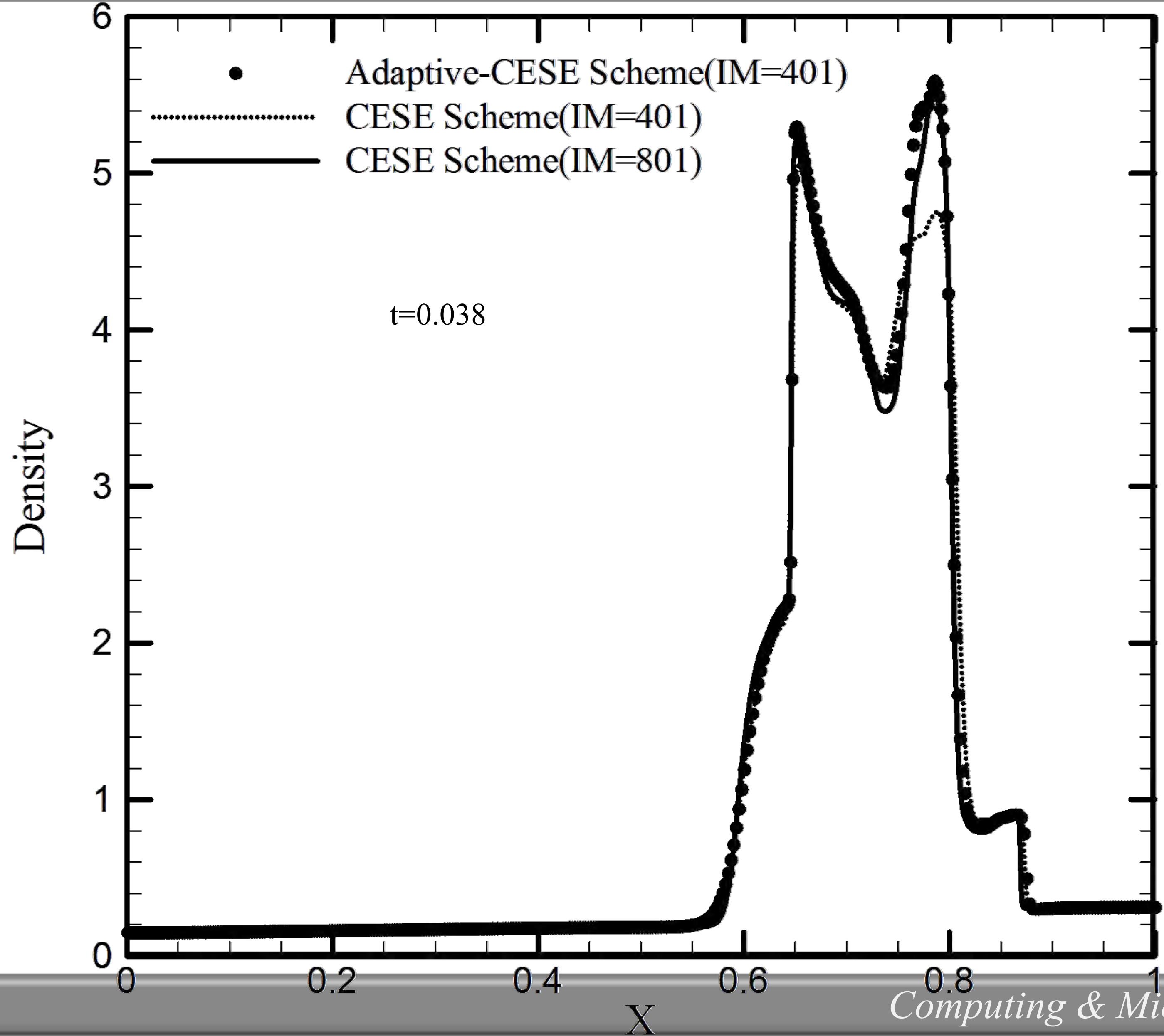
(2) The problem of two interaction blast waves

$$(\rho, u, p) = \begin{cases} (1, 0, 1000) & \text{if } x \le 0.1 \\ (1, 0, 0.01) & \text{if } 0.1 \le x \le 0.9 \\ (1, 0, 100) & \text{if } x \ge 0.9 \end{cases}$$



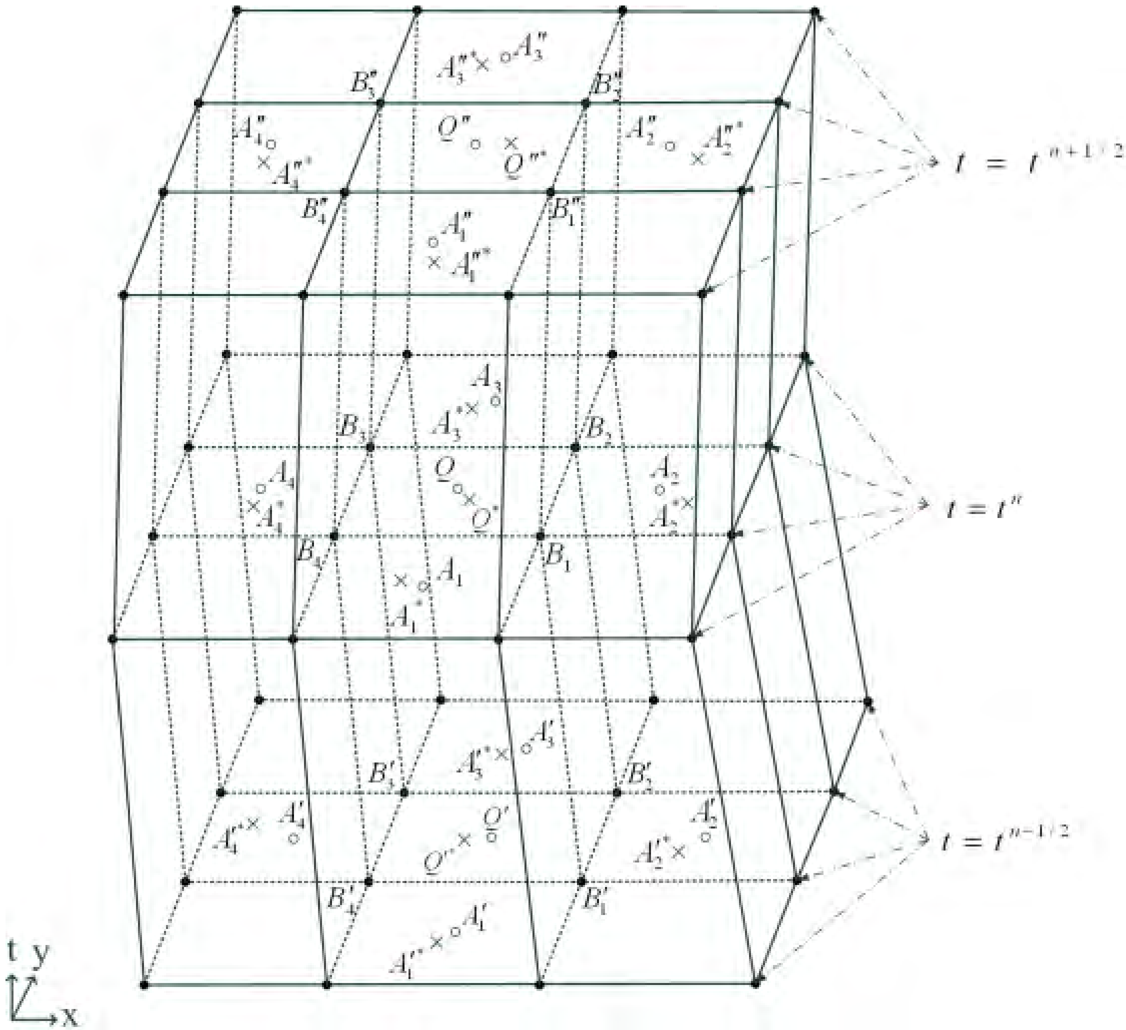


Shock tube problems (2)



Introduction of two-dimensional adaptive CESE method

Space-time domain



The space-time grid points at three adjacent time levels.

Solution element

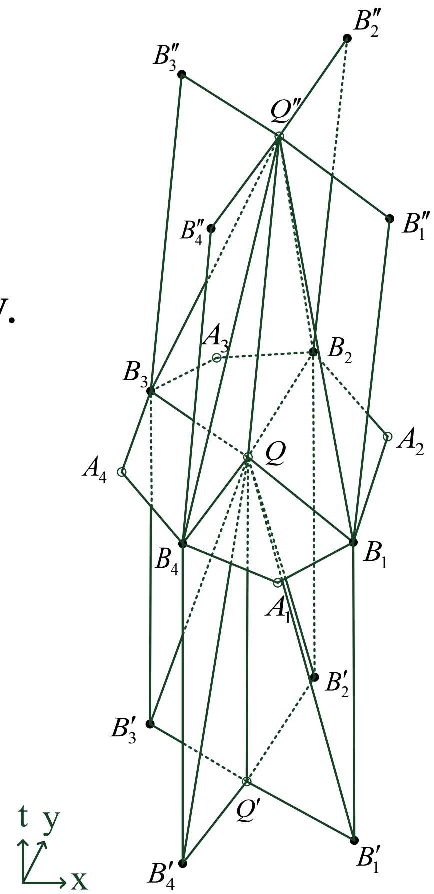
Flow properties are assumed continuities within SE. For any $(x,y,t)\in SE(Q^*)$ and any $m=1,2,3,4$, $u_m(x,y,t)$, $f_m(x,y,t)$, $g_m(x,y,t)$ and $\mathbf{h}_m(x,y,t)$, respectively, will be approximated by $u_m^*(x,y,t;Q^*)$, $f_m^*(x,y,t;Q^*)$, $g_m^*(x,y,t;Q^*)$ and $\mathbf{h}_m^*(x,y,t;Q^*)$ to be defined immediately.

Let

$$u_m^*(x,y,t;Q^*)=(u_m)_{Q^*}+(u_{mx})_{Q^*}(x-x_{Q^*})+(u_{my})_{Q^*}(y-y_{Q^*})+(u_{mt})_{Q^*}(t-t^n)$$

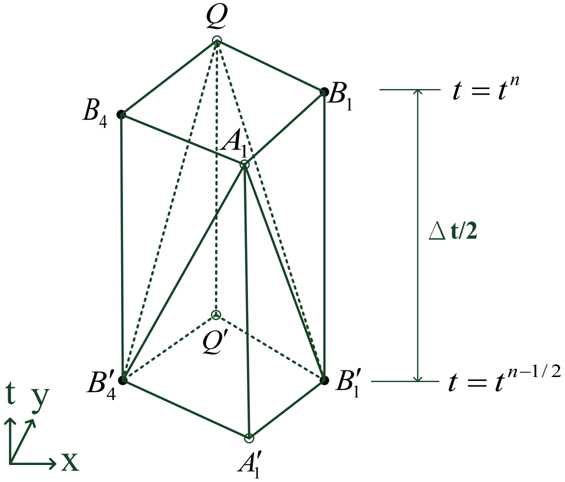
$$f_m^*(x,y,t;Q^*)=(f_m)_{Q^*}+(f_{mx})_{Q^*}(x-x_{Q^*})+(f_{my})_{Q^*}(y-y_{Q^*})+(f_{mt})_{Q^*}(t-t^n)$$

$$g_m^*(x,y,t;Q^*)=(g_m)_{Q^*}+(g_{mx})_{Q^*}(x-x_{Q^*})+(g_{my})_{Q^*}(y-y_{Q^*})+(g_{mt})_{Q^*}(t-t^n)$$

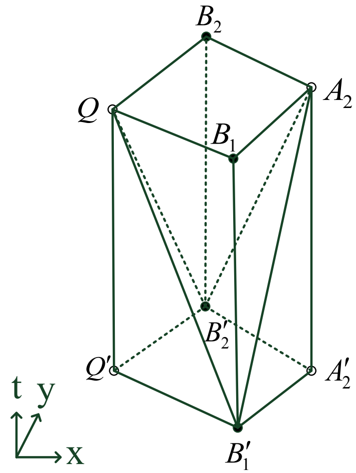


Solution element

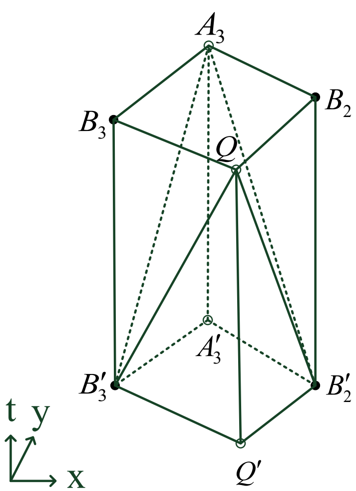
Basic conservation element



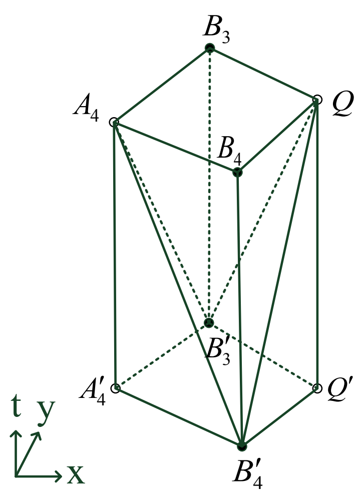
(a)



(b)



(c)



(d)

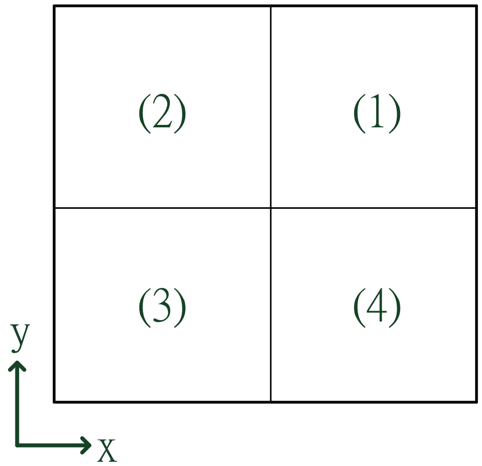
(a) The basic conservation element $CE^{(1)}(Q)$.

(b) The basic conservation element $CE^{(2)}(Q)$.

(c) The basic conservation element $CE^{(3)}(Q)$.

(d) The basic conservation element $CE^{(4)}(Q)$.

Two-dimensional Riemann problems



Initial conditions for the two-dimensional Riemann problems.

Numerical results

(a) adaptive mesh distribution

(b) adaptive CESE solution with 150x150 grid points

(c) the CESE solution with 150x150 grid points.

(d) the CESE solution with 400x400 grid points.

The spatial computation domain is defined by $[0,1]\times[0,1]$ and the initial conditions include four constant states.

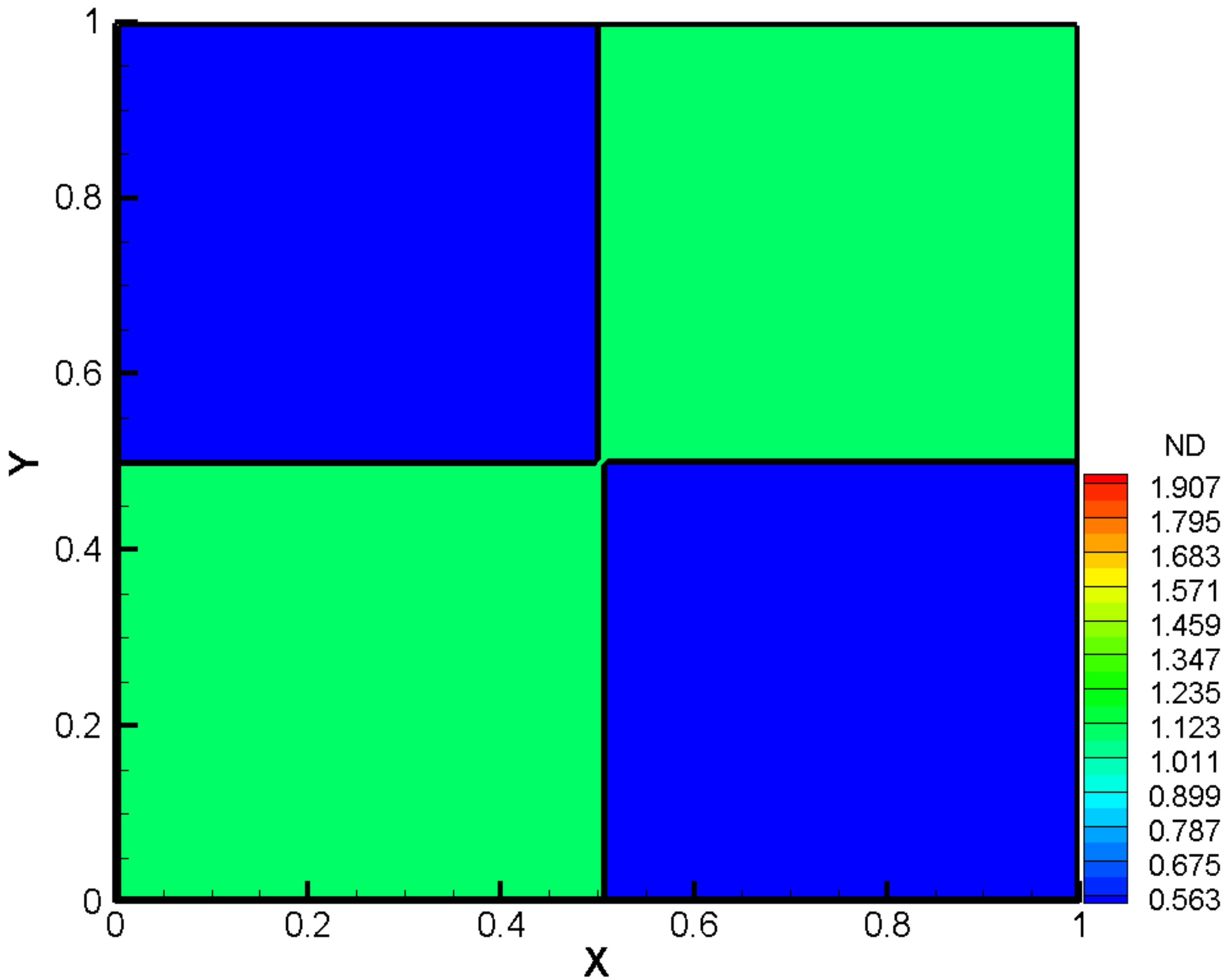
The four interfaces number are defined as $(\ell, k) = \{(1,2),(2,3),(3,4),(4,1)\}$

The forward and backward shock wave is denoted $S_{\ell k}^{+}$ and $S_{\ell k}^{-}$.

The forward and backward rarefaction wave is denoted $R_{\ell k}^{+}$ and $R_{\ell k}^{-}$.

The forward and backward slip line is denoted $J_{\ell k}^{+}$ and $J_{\ell k}^{-}$.

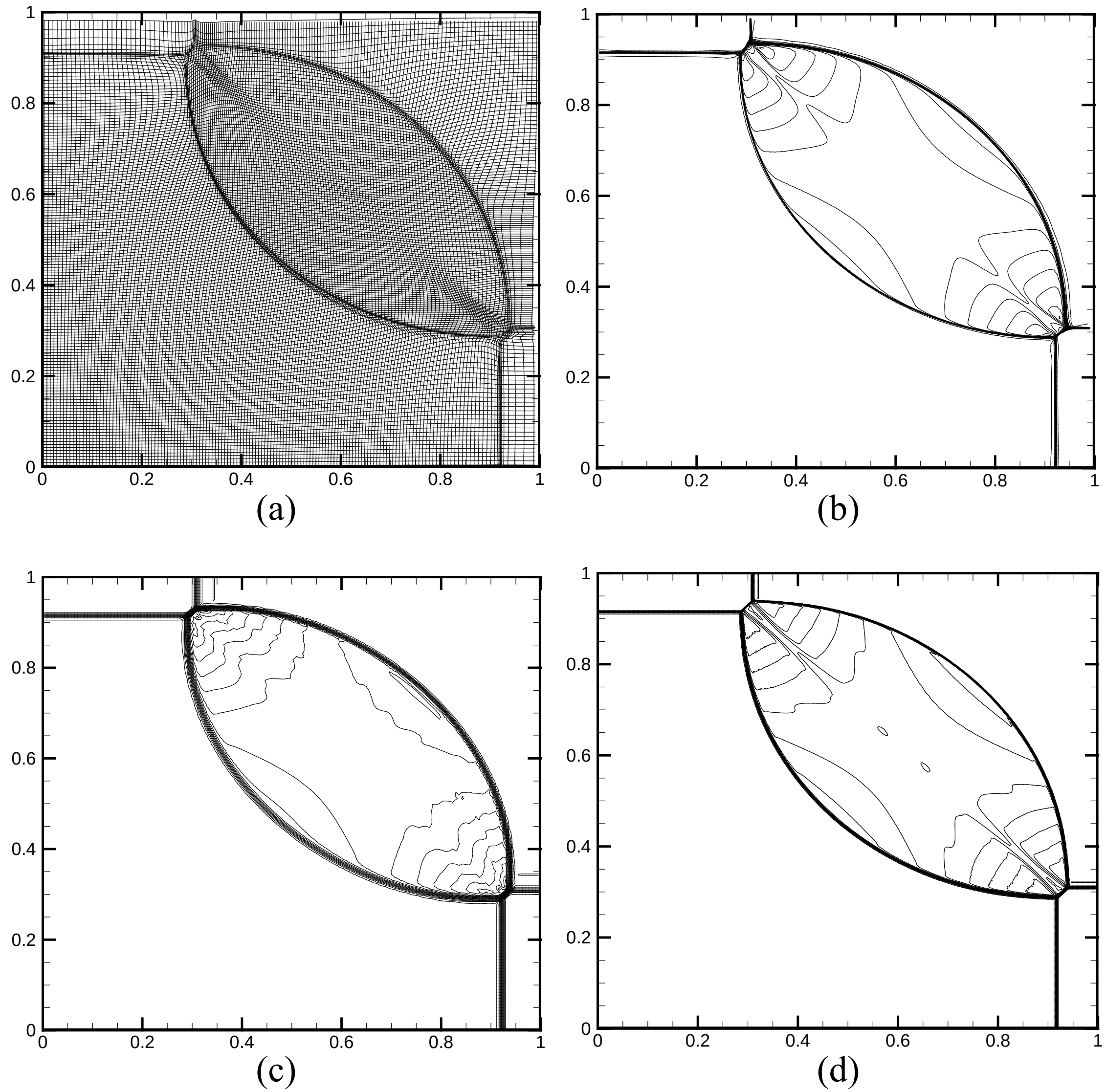
Riemann problems (1)
Pure shock waves interaction



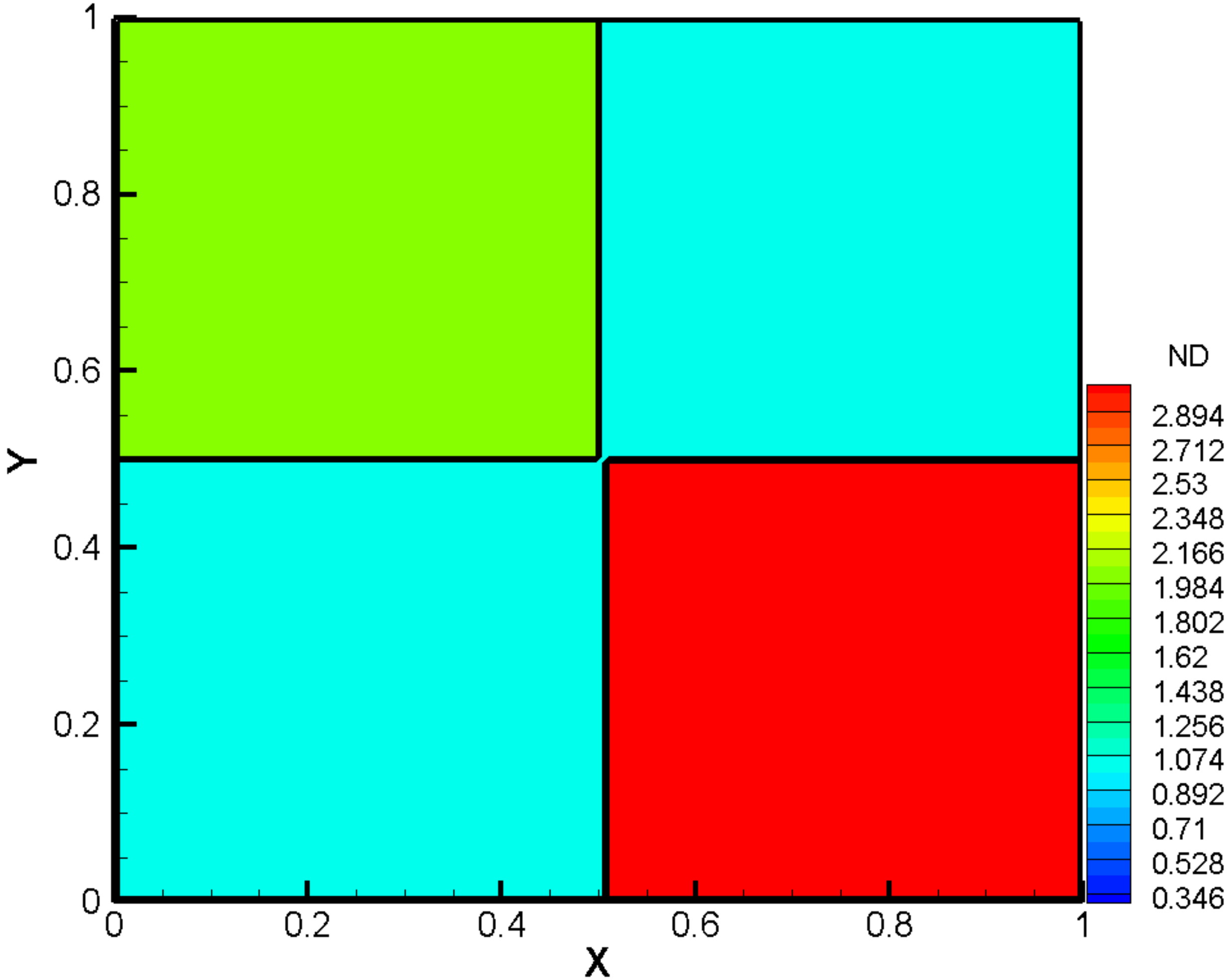
Sub-case $S_{12}^{+}S_{23}^{-}S_{34}^{+}S_{41}^{-}$:

$$(\rho, u, v, p) = \begin{cases} (1.1, 0, 0, 1.1) \text{-------------} (1) \\ (0.5065, 0.8939, 0, 0.35) \text{--}\ (2) \\ (1.1, 0.8939, 0.8939, 1.1) \text{--} (3) \\ (0.5065, 0, 0.8939, 0.35) \text{--} (4) \end{cases}$$

Riemann problems (1)
Pure shock waves interaction



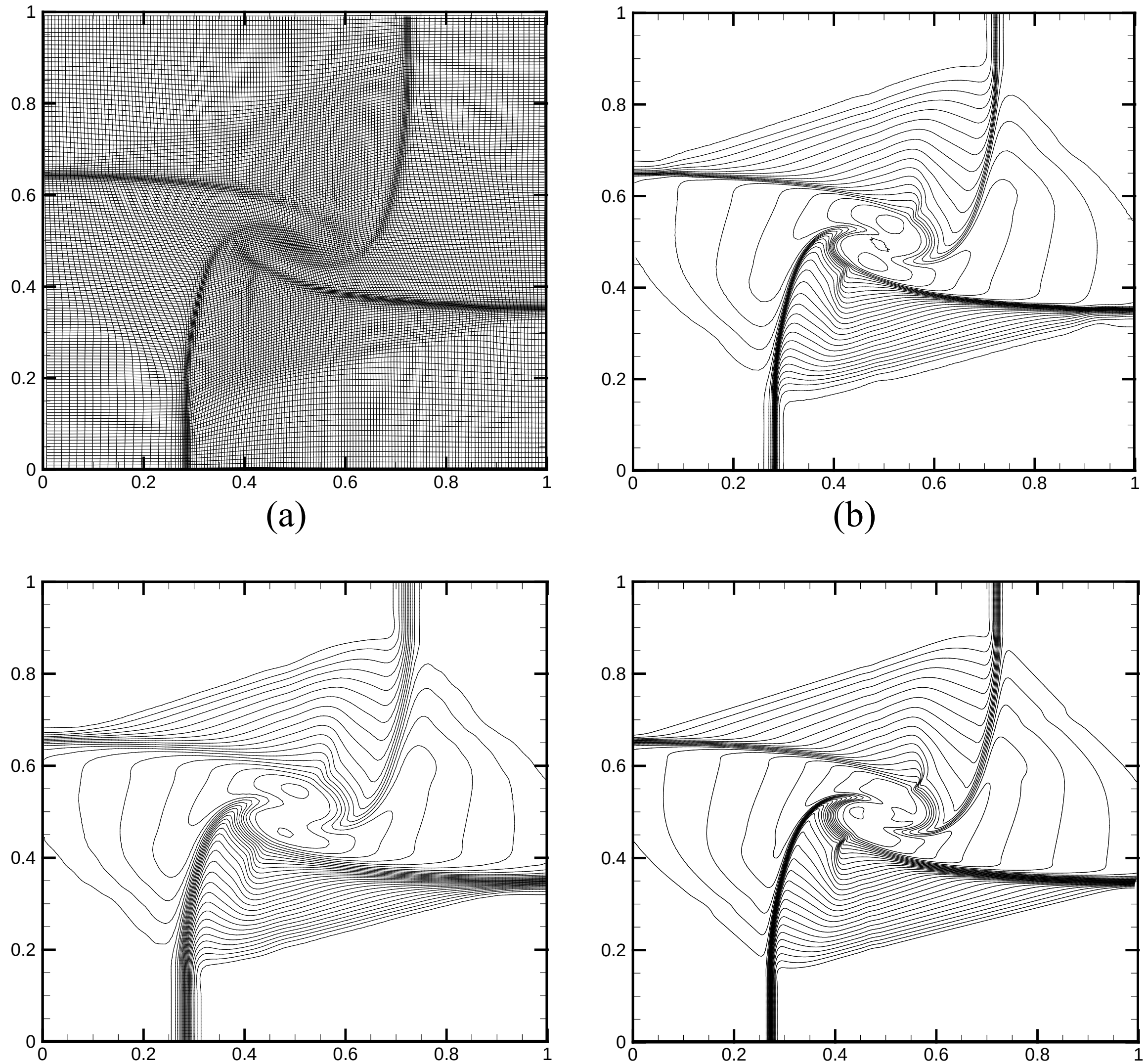
Riemann problems (2)
Pure slip line interaction



Sub-case $J_{12}^{-} J_{23}^{-} J_{34}^{-} J_{41}^{-}$:

$$(\rho, u, v, p) = \begin{cases} (1, 0.75, -0.5, 1) --- (1) \\ (2, 0.75, 0.5, 1) ---- (2) \\ (1, -0.75, 0.5, 1) --- (3) \\ (3, -0.75, -0.5, 1) -- (4) \end{cases}$$

Riemann problems (2)
Pure slip line interaction

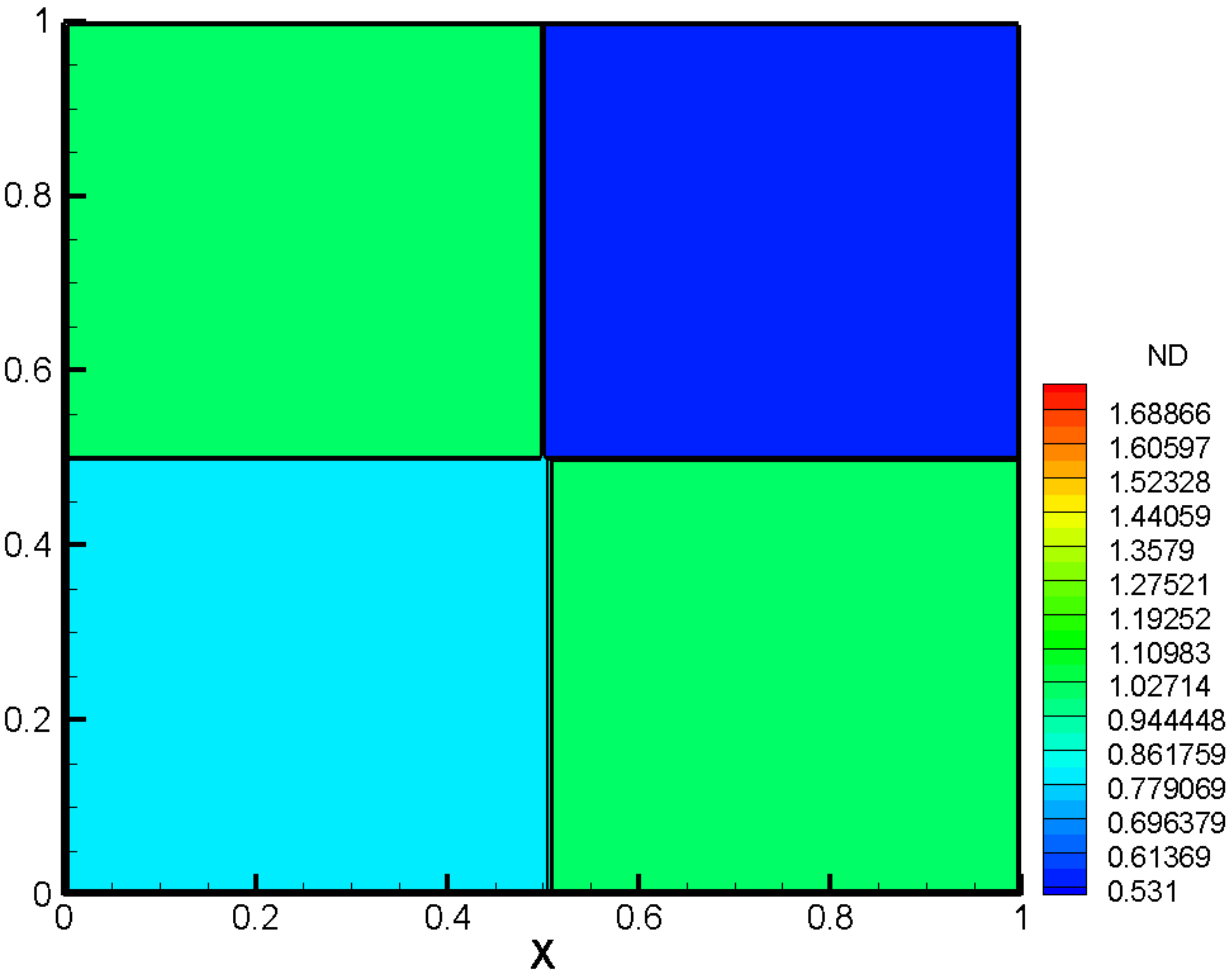


(a) (b)

(c) (d)

Riemann problems (3)

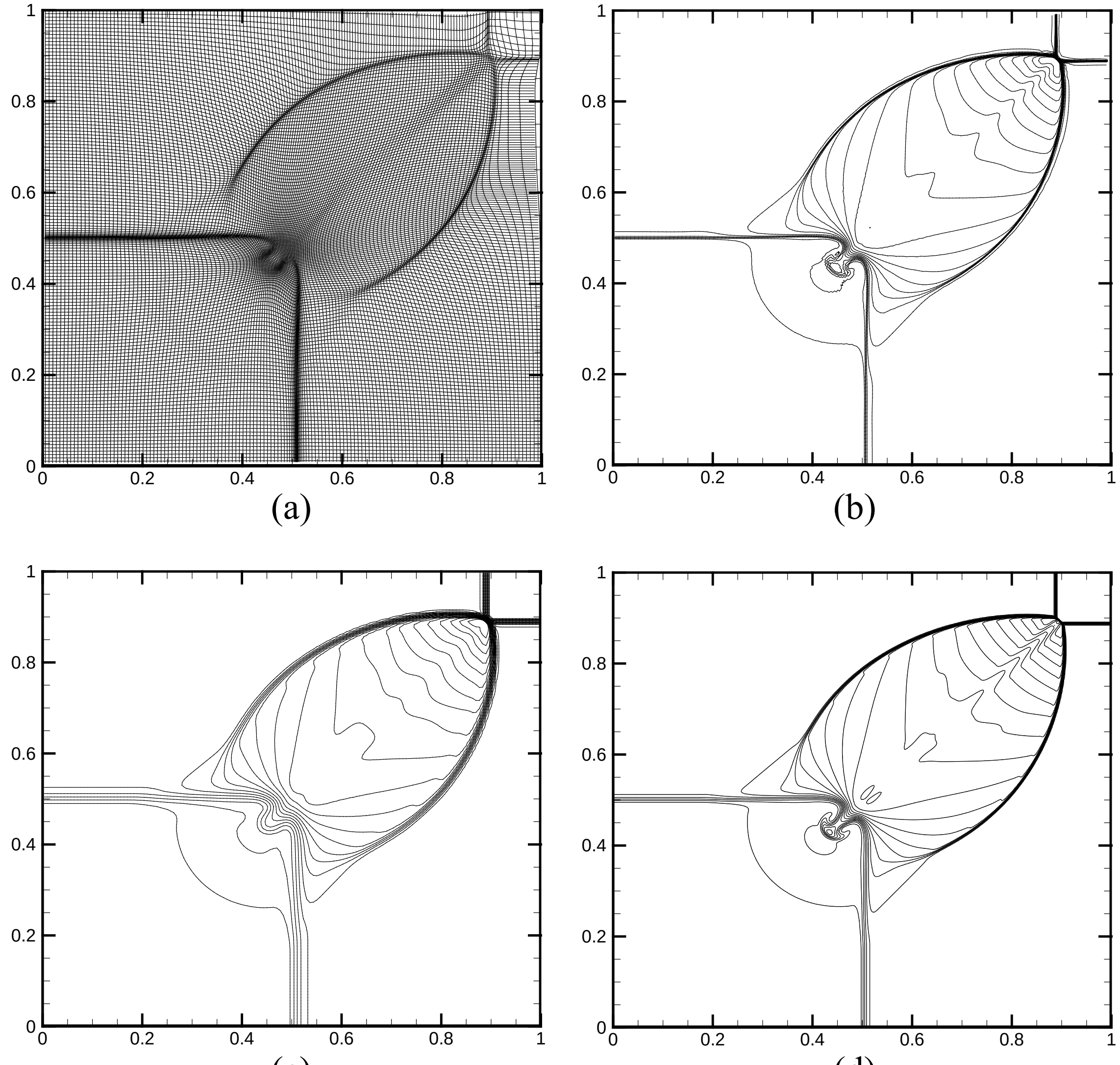
Interaction of multiple waves



Sub-case $S_{12}^{-}J_{23}^{-}J_{34}^{+}S_{41}^{+}$:

$$(\rho, u, v, p) = \begin{cases} (0.5313, 0, 0, 0.4) \text{--} (1) \\ (1, 0.7276, 0, 1) \text{----} (2) \\ (0.8, 0, 0, 1) \text{--------} (3) \\ (1, 0, 0.7276, 1) \text{----} (4) \end{cases}$$

Riemann problems (3)
Interaction of multiple waves



(a) (b) (c) (d)

Conclusion

(1) The CESE scheme can cover vast range of scales.

(2) The proposed adaptive CESE scheme in the current time-step is computed directly from the solutions obtained in the previous time-step without the need for extrapolation or interpolation.

(3) The adaptive CESE scheme captures the flow features with a significantly higher resolution than the original CESE solver.

(5) The adaptive CESE scheme (implemented on a coarse mesh) achieves the same (or better) resolution as the original stationary CESE solver on a fine mesh, but with a significantly lower computational cost.

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Progress is impossible without change, and those who cannot change their minds cannot change anything.

-- *George Bernard Shaw, Irish playwright*